Polyadic spaces and profinite monoids*

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Abstract. Hyperdoctrines are an algebraization of first-order logic introduced by Lawvere in [11]. In [9], Joyal defines a polyadic space as the Stone dual of a Boolean hyperdoctrine. He also proposed to recover a polyadic space from a simpler core, its Stirling kernel. We generalize this here in order to adapt polyadic spaces to certain classes of first-order theories. We will see how these ideas can be applied to give a correspondence between some first-order theories with a linear order symbol and equidivisible profinite semigroup with open multiplication. The inspiration comes from the paper [6] of van Gool and Steinberg, where model theory is used to study pro-aperiodic monoids.

Keywords: Categorical logic· Profinite monoids· Logic on words

1 Introduction

In [6], van Gool and Steinberg use model theory to study free pro-aperiodic monoids. The starting point is the Schützenberger-McNaughton-Pappert theorem, which establishes that languages on an alphabet A recognized by aperiodic monoids are those given by first-order sentences in Büchi's logic on finite words [13]. In category theory, a standard construction is to encode a monoid M as a functor $n \mapsto M^n$ whose domain is the category of finite linear orders and order-preserving maps. When $M = M_A$ is the free pro-aperiodic monoid on an alphabet A, we will see, using categorical logic, how this functor also represents Büchi's logic.

In Section 2, we present Boolean hyperdoctrines in parallel with their pointwise duals, polyadic Stone spaces. Boolean hyperdoctrines are an algebraization of classical first-order logic introduced by Lawvere in [11]. They are a way of representing a first-order theory by a functor from finite sets to Boolean algebras, sending a finite set X of variables to the Lindenbaum-Tarski algebra of formulas whose free variables are in X. In [9], Joyal defines a polyadic space as the pointwise Stone dual of a Boolean hyperdoctrine, so that the Lindenbaum-Tarski algebras are replaced by type spaces. Intuitively speaking, a polyadic space is thus a functor sending a finite set X to a space of X-pointed models. See also [12], which mentions the link with cylindric and polyadic algebras. We

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will start by associating two functors to each first-order theory: the first is the Boolean hyperdoctrine representing it and the second is the dual polyadic space. Afterwards, functors arising in this way are characterized by two simple axioms (Definitions 2.6 and 2.7).

Joyal also proposed to recover with a free construction a polyadic space from a simpler core, its Stirling kernel. In Section 3, we introduce different notions of kernels of polyadic spaces and the use of left Kan extension to reconstruct polyadic spaces from their kernels. For instance, suppose that a theory T has a partial order symbol.[1] Since models of T come with a partial order, it makes sense to speak of X-pointed models when X is a finite poset. Thus, one can replace finite sets in the domain of the polyadic space representing T by finite posets. The resulting functor will then be a "kernel" of the polyadic space of T. We give a number of examples of kernels, but a general framework is still to be elaborated.

In the last section, Section 4, we apply these ideas to monoids and languages. We will see that the functor $n \mapsto M_A^n$ referred to above, with M_A the free pro-aperiodic monoid, is a kernel of the polyadic space associated to Büchi's logic on words. In our first main result, Theorem 4.3, we generalize this situation and characterize the profinite semigroups S such that the functor $n \mapsto S^n$ represents a first-order theory as those whose multiplication is open and equidivisible. Profinite semigroups with these two properties are studied for instance by Almeida, Costa, Costa and Zeitoun in [1]. In their setting, equidivisibility alone is less well-behaved. We show here that openness and equidivisibility are natural from the viewpoint of logic since they each correspond to one of the axioms of polyadic spaces (Definition 2.7). We also characterize in our second main result, Theorem 4.8, first-order theories arising from a profinite monoid as those whose models can be concatenated with some natural constraints. Thus, these theories are very special.

Finite sets will be written as natural numbers, so that n can also denote the set $\{1, \ldots, n\}$. We write $\beta \colon \mathbf{Set} \to \mathbf{Stone}$ for the Stone-Čech compactification functor, which is left adjoint to the forgetful functor $\mathbf{Stone} \to \mathbf{Set}$. It is the usual Stone-Čech compactification restricted to discrete spaces. See [8, Chapter III, Section 2.1].

2 Hyperdoctrines and polyadic spaces

Boolean algebras are an algebraization of classical propositional logic. To get quantifiers in the picture, Lawvere remarked in [11] that they are adjoints to substitution of variables. We also get equality this way, and this gives a signature-independent algebraic account of classical first-order logic with equality. In this section, we explain how to arrive at this idea, using in parallel the dual notion of polyadic spaces introduced by Joyal in [9].

[1] More precisely, if there is a binary relational symbol $\leq$ subject to the axioms of partial orders.

2.1 Quantifiers as adjoints

Let T be a first-order theory. We associate to it a functor $D_T\colon \mathbf{FinSet} \to \mathbf{Bool}$ defined as follows:

1. On objects, D_T sends a finite set n to the n-Lindenbaum-Tarski algebra of the theory, i.e. the Boolean algebra of first-order formulas on n free variables modulo equivalence according to T. The set n is the "context" of the formulas in $D_T(n)$. We will write these variables as $x_1, \ldots, x_n$, or $y_1, \ldots, y_n$ if several contexts are present.
2. On morphisms, if $f\colon n \to m$ is a function, then $D_T(f)\colon D_T(n) \to D_T(m)$ sends a formula $\varphi(x_1, \ldots, x_n)$ to the formula $\varphi(y_{f(1)}, \ldots, y_{f(n)})$. That is, we substitute (in a capture-avoiding way) the variable x_i in $\varphi(x_1, \ldots, x_n)$ with the variable $y_{f(i)}$. In the special case where f is injective, this procedure only adds new variables to the context of the formula.

We will also use the notation $\varphi(\overline{x})$ for a formula on variables $\overline{x} = x_1, \ldots, x_n$, and write $\varphi(f(\overline{x}))$ for its image under $D_T(f)$.

A *Boolean hyperdoctrine* is a functor of the form D_T. A *polyadic space* is a functor $\mathbf{FinSet}^{\mathrm{op}} \to \mathbf{Stone}$ obtained by applying Stone duality pointwise to a hyperdoctrine. Let P_T be the polyadic space dual to D_T.

Remark 2.1. In model theory, $P_T(n)$ is usually called the space of (complete) *n-types* of T and is written $S_n(T)$ [7, Section 5.2]. It can be thought of as the set of n-pointed models of T modulo elementary equivalence. If $f\colon n \to m$ is a function, then $P_T(f)$ sends (the equivalence class of) an m-pointed model $(M, x_1, \ldots, x_m)$ to $(M, x_{f(1)}, \ldots, x_{f(n)})$.

Example 2.2. A simple example that one can keep in mind is $P_T(n) = X^n$ for X a set. If X is infinite, we actually need to take the Stone-Čech compactification $P_T(n) = \beta(X^n)$ to get a polyadic space (otherwise, we only get a polyadic *set*, see Subsection 3.3). The corresponding theory has one n-ary symbol for each subset of X^n, and theorems are formulas true for the obvious interpretation of these symbols on X.

Quantification Consider the following introduction-elimination rule for existential quantification: $\exists\overline{x} : \varphi(\overline{x}, \overline{y}) \vdash_{\overline{y}} \psi(\overline{y})$ if and only if $\varphi(\overline{x}, \overline{y}) \vdash_{\overline{x},\overline{y}} \psi(\overline{y})$. (See [10, Part 2, Section 1].) It is actually an adjunction formula, and more generally, if $f\colon n \rightarrowtail m$ is an injection, then the left adjoint of $D_T(f)$ is given by existential quantification over the variables in m which are not in the image of f. Symmetrically, universal quantification gives the right adjoint of $D_T(f)$. Notice that these adjoints are not morphisms of Boolean algebras in general.

Let us translate this in terms of P_T. Let $h\colon A \to B$ be a morphism of Boolean algebras. Then h has a left adjoint $h_*\colon B \to A$ if and only if the dual map $\tilde{h}\colon \tilde{B} \to \tilde{A}$ is open. In this case, h_* corresponds to taking the direct image by $\tilde{h}$. So the condition we obtain on P_T is that $P_T(f)$ is open for each injection $f\colon n \rightarrowtail m$ in $\mathbf{FinSet}$.

Example 2.3. Getting back to our example $P_T(n) = X^n$, we see that existential quantification corresponds to projection. For instance, the two projections $X^2 \to X$ correspond to quantification along the two variables.

Equality On the other hand, if $f\colon n \twoheadrightarrow m$ is a surjection, then $P_T(f)\colon P_T(m) \to P_T(n)$ is an injection whose image is the set of n-pointed models such that $x_i = x_j$ if $f(i) = f(j)$ (cf. Remark 2.1). Thus, $P_T(f)$ is open and dually, $D_T(f)$ has a left adjoint. The morphism $D_T(f)\colon D_T(n) \twoheadrightarrow D_T(m)$ realizes the quotient by the principal filter generated by $\bigwedge_{\substack{1\leq i,j\leq n\\ f(i)=f(j)}} [x_i = x_j]$.

Example 2.4. Binary equality is given by the quotient $2 \twoheadrightarrow 1$. In the case of $P_T(n) = X^n$, the corresponding map is the diagonal inclusion $X \rightarrowtail X^2$.

2.2 The Beck–Chevalley condition and quasi-pullbacks

If we look at logic only syntactically, one hidden aspect is that substitution commutes with all other constructions: Boolean operations, quantification and equality. In order to express the commutativity of substitution and existential quantification, let X, Y, Z be finite sets and let $f\colon X \to Z$ be a function. Let $\varphi(\overline{x}, \overline{y})$ be a formula in $D_T(X \sqcup Y)$. We have two ways of building $\exists \overline{y} : \varphi(f(\overline{x}), \overline{y})$, as illustrated below on the left.

$$\begin{array}{ccc} \exists\overline{y}\colon \varphi(\overline{x},\overline{y}) & \longmapsto & \exists\overline{y}\colon \varphi(f(\overline{x}),\overline{y}) \\ \uparrow & & \uparrow \\ \varphi(\overline{x},\overline{y}) & \longmapsto & \varphi(f(\overline{x}),\overline{y}) \end{array} \qquad \begin{array}{ccc} D_T(X) & \xrightarrow{D_T(f)} & D_T(Z) \\ \exists \uparrow\downarrow & & \exists \uparrow\downarrow \\ D_T(X \sqcup Y) & \xrightarrow{D_T(f\sqcup \mathrm{id}_Y)} & D_T(Z \sqcup Y) \end{array}$$

In the diagram on the right, the plain arrows are the images by D_T of the canonical maps and the dashed arrows labeled by $\exists$ are the left adjoints. Thus, the condition we are interested in is that the square with the dashed arrows commutes. Using Definition 2.5 below, this condition says that D_T sends pushouts of an injection ($X \rightarrowtail X \sqcup Y$) and an arbitrary function ($f\colon X \to Z$) to Beck–Chevalley squares.

Definition 2.5. *We say that the commutative square of posets* (1) *satisfies the* Beck–Chevalley condition *(or is a Beck–Chevalley square) if u and v have left adjoints u_* and v_* such that the square* (2) *commutes.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array} \quad (1) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u_*}\uparrow & & \uparrow{\scriptstyle v_*} \\ C & \xrightarrow{g} & D \end{array} \quad (2) \qquad \begin{array}{ccc} S(D) & \xrightarrow{v^{-1}} & S(B) \\ {\scriptstyle u^{-1}}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ S(C) & \xrightarrow{g^{-1}} & S(A) \end{array} \quad (3)$$

The dual of the Beck–Chevalley condition is given by the back and forth conditions of the duality for operators [4]. Let $S\colon \mathbf{Bool}^{\mathrm{op}} \to \mathbf{Stone}$ be the

Stone dualization functor. Then (1) is Beck–Chevalley if and only if its dual (3) is a *quasi-pullback* or *quasi-cartesian*, meaning that for each $b \in S(B)$ and each $c \in S(C)$ such that $f^{-1}(b) = g^{-1}(c)$, there is some $d \in S(D)$ such that $v^{-1}(d) = b$ and $u^{-1}(d) = c$.

More generally, D_T sends *any* pushout square to a Beck–Chevalley square.

2.3 Definitions

We arrive at an algebraic definition of hyperdoctrines and dually of polyadic spaces.

Definition 2.6. *A* (Boolean) hyperdoctrine *is a functor* $D\colon \mathbf{FinSet} \to \mathbf{Bool}$ *such that:*

1. *For each function f between finite sets, $D(f)$ has a left adjoint.*
2. *D sends pushout squares to Beck–Chevalley squares.*

Definition 2.7. *A* polyadic (Stone) space *is a functor* $P\colon \mathbf{FinSet}^{\mathrm{op}} \to \mathbf{Stone}$ *such that:*

1. *For each function f between finite sets, $P(f)$ is open.*
2. *P sends pushout squares in* **FinSet** *to quasi-pullbacks in* **Stone***.*

Functors of the form D_T are axiomatized by Definition 2.6 (see [16]) and functors of the form P_T are axiomatized by Definition 2.7. The map $T \mapsto D_T$ forgets which are the base symbols of the signature, so that all we can recover from D_T is the Morleyization of T.

Definition 2.8. *A morphism of hyperdoctrines is a natural transformation whose naturality squares satisfy the Beck–Chevalley condition. Dually, a morphism of polyadic spaces is a natural transformation whose naturality squares are quasi-cartesian.*

Definition 2.9. *A* model *of a hyperdoctrine D is a morphism from D to some hyperdoctrine of the form $n \mapsto \mathscr{P}(X^n)$ as in example 2.2. Dually, a model of a polyadic space P is a morphism of polyadic spaces $\beta(X^n) \to P(n)$.*

Models of D_T correspond to models of T: we interpret each relation definable over T as a subset of X. Since $\beta(X^n)$ is the free Stone space on X^n and using Proposition 2.11 below, models of a polyadic space P correspond to natural transformations $\alpha_n\colon X^n \to P(n)$ satisfying some weakening of Definition 2.8. This weakened condition says that if a n-tuple of the model satisfies some existential statement or an equality, then this is indeed concretely true in the model.

Definition 2.10. *In the commutative square on the left of* (4) *below, A and C are sets while B and D are Stone spaces with v continuous and open. We say that the square on the left of* (4) *is* weakly quasi-cartesian *if for each open $U \subseteq B$ and each $c \in C$ such that $g(c) \in v(U)$, there exists a witness $a \in A$ such that $f(a) \in U$ and $u(a) = c$.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array} \qquad\qquad \begin{array}{ccc} \beta(A) & \longrightarrow & B \\ {\scriptstyle \beta(u)}\downarrow & & \downarrow{\scriptstyle v} \\ \beta(C) & \longrightarrow & D \end{array} \tag{4}$$

Proposition 2.11. *In the same situation as in Definition 2.10, the square on the left of* (4) *is weakly quasi-cartesian if and only if the square on the right of* (4) *(in* **Stone***) is quasi-cartesian.*

Remark 2.12. Let $\alpha \colon X^n \to P(n)$ be a natural transformation. Intuitively, the naturality squares of α are weakly quasi-cartesian if X has "all the points it should have to be a model." If we strengthen this condition and ask that the naturality squares of α are quasi-cartesian, we recover the notion of *ω-saturated model.*

Remark 2.13. Using the Yoneda lemma, a natural transformation $X^n \to P(n)$ can be identified with a point of $P(X) = \lim_{n \subseteq X \text{ finite}} P(n)$. The X-based models of P are then identified with a subspace of $P(X)$. This is used in Rasiowa and Sikorski's proof of Gödel's completeness theorem in [14]: they prove that if X is countable and each $P(n)$ has a countable basis, then this subspace of models is comeager. This also yields the omitting types theorem [7, Theorem 6.2.1].

3 Kernels and C-adic spaces

3.1 Stirling kernels

The idea of Stirling kernels is due to Joyal (private communication). Let T be a first-order theory. Let $P_T[n] \subseteq P_T(n)$ be the clopen subspace of models pointed by n distinct points. Then we can reconstruct $P_T(n)$ from $P_T[n]$ with the finite coproduct

$$P_T(n) = \coprod_{R \in \mathrm{Cong}(n)} P_T[n/R], \tag{5}$$

where $\mathrm{Cong}(n)$ is the set of equivalence relations on n. Each term $P_T[n/R]$ is identified with the clopen subset of $P_T(n)$ of n-pointed models whose points are equal if and only if they are R-equivalent.

Since restricting a pointing by n distinct points along an injection gives again distinct points, $n \mapsto P_T[n]$ is a functor $\mathbf{FinSetInj}^{\mathrm{op}} \to \mathbf{Stone}$ where **FinSetInj** is the category of finite sets and injections. This functor is called the *Stirling kernel* of $P_T(n)$, because the Stirling numbers of the second kind are the coefficients of $P_T[k]$ in the formula (5) (there is one copy of $P_T[k]$ for each equivalence relation of index k on n).

Another way of expressing the formula (5) is to say that $P_T(n)$ is the left Kan extension of $P_T[n]$ along the inclusion $\mathbf{FinSetInj}^{\mathrm{op}} \rightarrowtail \mathbf{FinSet}^{\mathrm{op}}$. For an introduction to Kan extensions, see [2, Section 3.7]. This also reconstructs the functoriality of $P_T(-)$. More concretely, let $f \colon n \to m$ be any function and let

R be an equivalence relation on m. Let $f^{-1}(R)$ be the equivalence relation on n induced by R and f and let $\tilde{f}\colon n/f^{-1}(R) \to m/R$ be the corestriction of f. Then $P_T[m/R]$ is sent into $P_T[n/f^{-1}(R)]$ by $P_T(f)$ and this restriction coincides with $P_T[\tilde{f}]$.

In Theorem 3.4 below, we identify the conditions for a functor $\mathbf{FinSetInj}^{\mathrm{op}} \to \mathbf{Stone}$ to be a Stirling kernel. This provides another algebraization of first-order logic with equality. In some sense, this tells us that equality can be "freely adjoined".

In Definition 2.7, the first condition says that $P(f)$ is open and this is transferred to the Stirling kernel since $P[f]$ is a restrictions of $P(f)$ to a clopen subset. On the other hand, the second condition cannot be transferred so easily since the category **FinSetInj** does not admit pushouts. Let us give an example to illustrate that point.

Example 3.1. Consider the diagram $1 \leftarrow 0 \rightarrow 1$ in **FinSetInj**. The two minimal cocones over it are $1 \rightarrow 2 \leftarrow 1$ (the pushout in **FinSet**) and $1 \rightarrow 1 \leftarrow 1$. Since there is no injection $2 \rightarrow 1$, the pushout in **FinSet** is not a pushout in **FinSetInj** anymore. And indeed, the square (6) below is not a quasi-pullback: given two 1-pointed models (M, x) and (M', x') with $M \equiv M'$, it is possible that the only way to amalgamate them into a single 2-pointed model (N, a, b) with $(N, a) \equiv (M, x)$ and $(N, b) \equiv (M', x')$ is to take $a = b$. This means that instead of finding a point of $P_T[2]$ in the upper left corner of (6), we find a point of $P_T[1]$ in the upper left corner of (7).

$$\begin{array}{ccc} P_T[2] & \longrightarrow & P_T[1] \\ \downarrow & & \downarrow \\ P_T[1] & \longrightarrow & P_T[0] \end{array} \quad (6) \qquad \begin{array}{ccc} P_T[1] & \longrightarrow & P_T[1] \\ \downarrow & & \downarrow \\ P_T[1] & \longrightarrow & P_T[0] \end{array} \quad (7) \qquad \begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \vdots\, {\scriptstyle u} \\ c & \overset{v}{\dashrightarrow} & d \end{array} \quad (8)$$

In order to take this into account, we need the following definitions.

Definition 3.2. *A functor $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ has the* amalgamation property *if for any diagram $(f\colon a \to b, g\colon a \to c)$ in $\mathbf{C}$, for any $x \in P(b)$, $y \in P(c)$ such that $P(f)(x) = P(g)(y)$, there exists a cocone as in (8) and a witness $z \in P(d)$ such that $P(u)(z) = x$ and $P(v)(z) = y$. Notice that d can also depend on x and y. A natural transformation is said to have the amalgamation property if each naturality square is quasi-cartesian. We use the same vocabulary for functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Stone}$. A natural transformation $\alpha\colon P \to Q$ has the* weak amalgamation property *if its naturality squares are weakly quasi-cartesian (Definition 2.10).*

We are now ready to generalize Definition 2.7 appropriately.

Definition 3.3. *Let $\mathbf{C}$ be a small category. A $\mathbf{C}$-adic* (Stone) space *is a functor $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Stone}$ with the amalgamation property and such that $P(f)$ is open for each arrow f in $\mathbf{C}$. A morphism of $\mathbf{C}$-adic spaces is a natural transformation with the amalgamation property.*

We will refer to $\mathbf{C}$ as the category of contexts. If $\mathbf{C}$ has pushouts, we recover the notion of Boolean hyperdoctrine since any cocone as in (8) factors through the pushout, and the amalgamation property says that pushout squares are sent to quasi-pullbacks. In particular, we get polyadic spaces with $\mathbf{C} = \mathbf{FinSet}$.

Theorem 3.4. *The functor* $[\mathbf{FinSetInj}^{\mathrm{op}}, \mathbf{Stone}] \to [\mathbf{FinSet}^{\mathrm{op}}, \mathbf{Stone}]$ *given by left Kan extension along* $\mathbf{FinSetInj}^{\mathrm{op}} \rightarrowtail \mathbf{FinSet}^{\mathrm{op}}$ *restricts to an equivalence between the category of* $\mathbf{FinSetInj}$*-adic spaces and the category of polyadic spaces.*

Given a first-order theory T, models of T can be defined in terms of $P_T[-]$ as a set X equipped with a natural transformation $\{\text{injections } n \rightarrowtail X\} \to P_T[n]$ with the weak amalgamation property.

3.2 Other kernels

If we restrict our attention to smaller classes of first-order theories, it may be possible to consider more specialized kernels. We will only give here examples of kernels and no general definition.

Suppose T is a first-order theory with a distinguished linear order symbol. Let Δ_+ be the category of (possibly empty) finite linear orders and order-preserving maps. Then $P_T\langle n\rangle := \{\text{models of } T \text{ pointed by } n \text{ increasing points}\} \subseteq P_T(n)$ defines a functor $\Delta_+^{\mathrm{op}} \to \mathbf{Stone}$. Left Kan extension along the forgetful functor $\Delta_+ \to \mathbf{FinSet}$ yields an equivalence from the category of Δ_+-adic spaces to the category of polyadic spaces equipped with a linear order relation (morphisms need to respect this relation). Combining this with the idea of Stirling kernels, this category is again equivalent to the category of $\Delta_{+,\mathrm{inj}}$-adic spaces, where $\Delta_{+,\mathrm{inj}}$ is the category of finite linear orders and injective order-preserving maps.

An order is *bounded* if there is a greatest and a least element. Let Δ_{bound} be the category of finite bounded linear orders and bound-preserving order-preserving maps. Let $\Delta_{\mathrm{bound,inj}}$ be the subcategory of Δ_{bound} with only the injective morphisms. Then the Δ_{bound}-adic and $\Delta_{\mathrm{bound,inj}}$-adic spaces both correspond to first-order theories with a bounded linear order. The diagram (9) below summarizes the situation. Translation across different categories of contexts is done through left Kan extension along the arrows.

$$\begin{array}{ccccl}
\mathbf{FinSet} & \longleftarrow & \mathbf{FinSetInj} & & \text{first order theories} \\
\uparrow & & \uparrow & & \\
\Delta_+ & \longleftarrow & \Delta_{+,\mathrm{inj}} & & \text{with a linear order} \\
\uparrow & & \uparrow & & \\
\Delta_{\mathrm{bound}} & \longleftarrow & \Delta_{\mathrm{bound,inj}} & & \text{with a bounded linear order}
\end{array} \tag{9}$$

As in the case of Stirling kernels, models are the intuitive ones: objects X with a natural transformation from $\{\text{maps } n \to X\}$ to $P(n)$ with the weak amalgamation property, where "objects" and "maps" are interpreted according to the situation. In general, a model of a **C**-adic space P is an ind-object $X \in \mathbf{Ind}(\mathbf{C})$ equipped with a natural transformation $X \to P$ with the weak amalgamation property. For an introduction to ind-objects, see [8, Chapter VI, Section 1].

Example 3.5. Here are some more examples of kernels. Items 5 and 6 will be used in Section 4.

1. The same thing works for ordered sets, graphs, pointed sets, sets with an equivalence relation, various notions of trees and forests,[2] etc.
2. If T is an algebraic theory and $\mathbf{Alg}^T_{\mathrm{fp}}$ is the category of finitely presented T-algebras, then $\mathbf{Alg}^T_{\mathrm{fp}}$-adic spaces correspond to first-order theories with a distinguished interpretation of T.
3. A $\mathbf{FinSet} \times \mathbf{FinSet}$-adic space is a theory in 2-sorted first order logic, or in other words a first-order theory with a distinguished unary relational symbol R (one sort is the interpretation of R and the other one is its complement). Taking the left Kan extension along $+\colon \mathbf{FinSet} \times \mathbf{FinSet} \to \mathbf{FinSet}$ forgets the distinguished unary relational symbol.
4. A pointed bounded linear order is a pair of bounded linear orders, so a $\Delta^2_{\mathrm{bound}}$-adic space is a Δ_{bound}-adic space with a distinguished constant. The translation is made via a left Kan extension along the functor $\boxplus\colon \Delta^2_{\mathrm{bound}} \to \Delta_{\mathrm{bound}}$ joining two bounded linear orders by gluing their endpoints.
5. If $P, Q\colon \Delta^{\mathrm{op}}_{\mathrm{bound}} \to \mathbf{Stone}$ are two Δ_{bound}-adic spaces, then $(a,b) \mapsto P(a) \times Q(b)$ is the $\Delta^2_{\mathrm{bound}}$-adic space whose models are bounded linear orders with a distinguished point, a P-structure on the lower half and a Q-structure on the upper half.
6. If P is a Δ_{bound}-adic space, then $(a,b) \mapsto P(a \boxplus b)$ is the $\Delta^2_{\mathrm{bound}}$-adic space whose models are pointed models of P. Actually, precomposition by $\boxplus$ defines a functor from Δ_{bound}-adic spaces to $\Delta^2_{\mathrm{bound}}$-adic spaces and it is the right adjoint to left Kan extension along $\boxplus$.
7. On the other hand, it is *false* that if P is a **FinSet**-adic space, then $(a,b) \mapsto P(a+b)$ is the $\mathbf{FinSet}^2$-adic space whose models are models of P with a distinguished subset, as it was the case for Δ_{bound}.

3.3 Some properties of C-adic spaces

Instead of **C**-adic spaces, we can consider **C**-adic *sets*. They are functors $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ with the amalgamation property. Under some hypothesis on **C**, they can be compactified to give **C**-adic spaces.

Proposition 3.6. *Let* **C** *be a small category. For any two arrows* $f\colon a \to b$*,* $g\colon a \to c$ *with a common domain, let* $\mathbf{C}_{f,g}$ *be the category of cocones over the diagram composed of* f *and* g*. Suppose that for each* $\mathbf{C}_{f,g}$*, there is a weakly initial finite set of objects* $T_{f,g}$*, i.e. such that each other object admits an arrow from an element of* $T_{f,g}$*. Then post-composition by* $\beta\colon \mathbf{Set} \to \mathbf{Stone}$ *gives a functor from* **C***-adic sets to* **C***-adic Stone spaces.*

[2] Trees and forests are considered as special ordered sets for the notion to be first-order.

Remark 3.7. The hypothesis of Property 3.6 above is also the condition for the duals of $\mathbf{C}$-adic spaces to be a multisorted algebraic variety, with one sort for each object of $\mathbf{C}$. A consequence of this algebraic presentation is that products of hyperdoctrines are computed pointwise and dually coproducts of $\mathbf{C}$-adic spaces are computed pointwise. Another consequence is that subalgebras are again hyperdoctrines, but this is true more generally as seen in the proposition below.

Proposition 3.8. *Let $\mathbf{C}$ be a small category. Let $P, Q \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Stone}$. Suppose P is a $\mathbf{C}$-adic space. Let $\alpha \colon P \to Q$ be a natural transformation that is pointwise surjective. If α has the amalgamation property, then Q is a $\mathbf{C}$-adic space.*

Remark 3.9. A *polyadic subspace* of a polyadic space P is a morphism of polyadic spaces $Q \to P$ which is pointwise injective. If $\mathbf{C}$ has an initial object 0, then polyadic sub-spaces of P are classified by closed subsets of $P(0)$. In particular, P cannot be written as a non-trivial coproduct if and only if $P(0) = 1$, i.e. if the theory is "complete." If P is a $\mathbf{C}$-adic *set* instead of a $\mathbf{C}$-adic Stone space, then we can maximally decompose P as a coproduct indexed by points of $P(0)$.

Gödel's completeness We can generalize the statement of Gödel's completeness theorem to $\mathbf{C}$-adic spaces. We will say that $\mathbf{C}$ satisfies Gödel's completeness theorem if for any $\mathbf{C}$-adic space P, for any $X \in \mathbf{Ind}(\mathbf{C})$ and any natural transformation $\alpha \colon X \to P$, we can extend this "pre-model" to a model, i.e. find an ind-object $Y \in \mathbf{Ind}(\mathbf{C})$, a natural transformation $\tilde{\alpha} \colon Y \to P$ with the weak amalgamation property and a natural transformation $f \colon X \to Y$ such that $\alpha = \tilde{\alpha} \circ f$. When $\mathbf{C}$ has pushouts, the proof of the classical Gödel theorem with Henkin's models can be adapted. But this is not enough to cover all the cases where it is true, since it can sometimes be deduced from other categories of contexts through the use of kernels. For instance, none of the categories of Diagram (9) admit pushouts except for $\mathbf{FinSet}$, but Gödel's completeness theorem for these categories can be deduced from $\mathbf{FinSet}$.

When Gödel's completeness theorem holds, it allows for a description of morphisms in terms of models. Given a $\mathbf{C}$-adic space P, we say that a family of clopen subsets of the spaces $P(c)$ is *generating* if for each $c \in \mathbf{C}$, each clopen subset of $P(c)$ can be expressed from the given family of clopen subsets using the Boolean operations together with direct and inverse images by maps of the form $P(f)$. If $\mathbf{C} = \mathbf{FinSet}$ and if $P = P_T$ is the polyadic space associated to a first-order theory T, then the clopen subsets corresponding to the symbols of the signature of T are generating.

Proposition 3.10. *Let $\mathbf{C}$ be a category satisfying Gödel's completeness theorem. Let P and Q be two $\mathbf{C}$-adic spaces. Then a morphism $P \to Q$ is given by a way of transforming, for any $X \in \mathbf{Ind}(\mathbf{C})$, P-structures on X into Q-structures on X such that for any clopen subset $\varphi \subseteq Q(c)$, there exists a clopen subset $F(\varphi) \subseteq P(c)$ defining the same subset of $X(c)$ for each P-model X. Moreover, it is enough to check this condition on a generating family of clopen subsets.*

Remark 3.11. This way of viewing morphisms of polyadic spaces respects composition. In particular, an isomorphism $P \cong Q$ is given by the data of a bijection between P-structures on X and Q-structures on X for each ind-object X, such that each subset of $X(c)$ definable over Q is definable over P (uniformly in X) and conversely. Beth's definability theorem, which is a consequence of Gödel's completeness theorem in this setting, says that this converse is automatically satisfied: any morphism $P \to Q$ of $\mathbf{C}$-adic spaces inducing a bijection between P-models and Q-models is an isomorphism.

4 Application to logic on words

4.1 Correspondence between theories and monoids

As we saw in Subsection 3.2, a first-order theory with a bounded linear order symbol is a special functor $\varDelta_{\text{bound}}^{\text{op}} \to \mathbf{Stone}$. Looking at a bounded linear order as a lattice, the duality between finite lattices and finite posets specializes to a duality between $\varDelta_{\text{bound}}$ and $\varDelta_+$. Hence, functors $\varDelta_{\text{bound}}^{\text{op}} \to \mathbf{Stone}$ correspond to functors $\varDelta_+ \to \mathbf{Stone}$ by precomposing with this equivalence. Functors $\varDelta_+ \to \mathbf{Stone}$ can also be used to encode profinite monoids S, meaning that S is also a Stone space with continuous multiplication. The functor associated to S will be called P_S. It sends n to S^n, the injection $0 \to 1$ to the neutral element $e\colon 1 \cong S^0 \to S^1$ and the unique surjection $2 \to 1$ to the multiplication $S^2 \to S$.

Before we continue, we should introduce some notation. When we want to emphasize the ambient category, we will write the object of $\varDelta_+$ with n elements as $\underline{n}$, and the object of $\varDelta_{\text{bound}}$ with n elements as $\underbracket{n}$. For convenience, we will also write $\underbracket{n} \mapsto P_S(\underbracket{n})$ for the functor obtained by precomposing $\underline{n} \mapsto P_S(\underline{n})$ with the equivalence $\varDelta_{\text{bound}}^{\text{op}} \cong \varDelta_+$. We will write $\underline{n} + \underline{m}$ for the linear order obtained by concatenating $\underline{n}$ and $\underline{m}$. The dual of this operation is $\underbracket{n} \boxplus \underbracket{m} = \underbracket{n - 1 + m}$, obtained by merging the last point of $\underbracket{n}$ with the first point of $\underbracket{m}$.

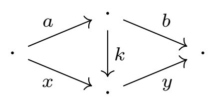
The condition on P_S to encode a monoid is that it is *monoidal*, meaning that $P_S(\underline{0}) = 1$ and that it comes equipped with an isomorphism $P_S(\underline{n}+\underline{m}) \cong P_S(\underline{n}) \times P_S(\underline{m})$ satisfying with some commutativity conditions, or dually an isomorphism $P_S(\underbracket{n} \boxplus \underbracket{m}) \cong P_S(\underbracket{n}) \times P_S(\underbracket{m})$. If we replace the duality $\varDelta_+ \cong \varDelta_{\text{bound}}^{\text{op}}$ with the duality $\varDelta_{+,\text{surj}} \cong \varDelta_{\text{bound,inj}}^{\text{op}}$, we get the same thing but with semigroups instead of monoids. Left Kan extension along $\varDelta_{+,\text{surj}} \rightarrowtail \varDelta_+$ freely adjoins a neutral element. Note that a monoidal functor is obtained by adding some structure to a functor, so that we will speak of monoidal functor structures.

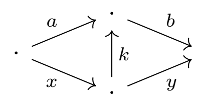
The two questions raised by this connection are:

1. When does a monoid correspond to a theory with a bounded linear order symbol? More precisely, under which conditions on S is P_S a $\varDelta_{\text{bound}}$-adic space? (Theorem 4.3.)
2. When does a theory with a bounded linear order symbol correspond to a monoid? More precisely, given a $\varDelta_{\text{bound}}$-adic space P, how can we understand a monoidal functor structure on P from the viewpoint of logic? (Theorem 4.8.)

From monoids to theories Let S be a semigroup. The amalgamation property of $P_S\colon \underline{n} \mapsto S^n$ applied to the diagram $\underline{2} \to \underline{1} \leftarrow \underline{2}$ gives the definition below. The three cases correspond to the three minimal cones over $\underline{2} \to \underline{1} \leftarrow \underline{2}$ in $\Delta_{+,\mathrm{surj}}$.

Definition 4.1. *A monoid or semigroup S is* equidivisible *if for any $a, b, x, y \in S$ such that $ab = xy$, either $(a, b) = (x, y)$ or there is a $k \in S$ such that $ak = x$ and $ky = b$, or $xk = a$ and $kb = y$. In the case of monoids, the definition can be simplified by taking $k = e$ when $(a, b) = (x, y)$.*





As we see in the next proposition, equidivisibility is enough to recover the full amalgamation property.

Proposition 4.2. *Let S be a semigroup. Then P_S has the amalgamation property if and only if S is equidivisible. If S is a monoid, then P_S has the amalgamation property if and only if S is obtained by freely adjoining an identity to an equidivisible semigroup.*

As a consequence, we obtain the following theorem.

Theorem 4.3. *Let S be a profinite semigroup. Then P_S is a $\Delta_{\mathrm{bound,inj}}$-adic space if and only if S is equidivisible and its multiplication is open.*

Let S be a profinite monoid. Then P_S is a Δ_{bound}-adic space if and only if S is obtained from a profinite equidivisible semigroup with open multiplication by freely adjoining an isolated neutral element.

Let S be a profinite monoid as in Theorem 4.3. Here is a description of what is a P_S-model, or S-model for short.

Proposition 4.4. *Let X be a bounded linear order. Think of X as a category with* $\mathrm{Hom}(x, y) = 1$ *if $x \leq y$ and* 0 *otherwise. Think of S as a category with one object. Then a structure of S-model on X is given by a functor $F\colon X \to S$ such that, writing $F(x, y)$ for the image of the unique arrow $x \to y$ when it exists, we have $F(x, y) = e \iff x = y$ and for any clopen subset $U, V \subseteq S$ and any $x \leq y$ in X,*

$$F(x, y) \in UV \iff \exists z \in [x, y] : F(x, z) \in U \wedge F(z, y) \in V. \tag{10}$$

Remark 4.5. If we replace (10) by the following, we get the notion of ω-saturated model (cf. Remark 2.12).

$$F(x, y) = uv \iff \exists z \in [x, y] : F(x, z) = u \wedge F(z, y) = v.$$

Remark 4.6. There is also a natural axiomatization of this theory. Each clopen subset $U \subseteq S$ corresponds to a binary relational symbol on X that can be satisfied by two points x, y only if $x \leq y$. We write $U(x, y)$ for this symbol. Here are the axioms.

1. All the axioms for bounded linear orders.
2. $U(x, y) \implies x \leq y$
3. $[U \cap V](x, y) \iff U(x, y) \wedge V(x, y)$ and similarly for all Boolean operations.
4. $\{e\}(x, y) \iff x = y$
5. $[UV](x, y) \iff \exists z : U(x, z) \wedge V(z, y)$

The set of axioms 3 ensures that each couple of points determines a prime filter on the dual Boolean algebra of S and thus a point of S. Predicates on n ordered free variables (i.e. under the condition $x_1 \leq \cdots \leq x_n$) are identified to clopen subsets of S^{n+1}. An n-type of this theory can be described as an ordering of the n variables together with an element $(a_0, \ldots, a_n) \in S^{n+1}$, subject to the relation that if $a_i = e$, then we can exchange the ith variable and the $(i+1)$th variable in the ordering without modifying the n-type.

From theories to monoids Let's now turn to the second question: Given a Δ_{bound}-adic space P, how can we understand a monoidal functor structure on P from the viewpoint of logic? We will first study the structure transferred from S to P_S. Let S be a profinite monoid such that P_S is a Δ_{bound}-adic space. Proposition 4.7 below shows that the monoid structure of S underlies a monoid structure on the class of S-models.

Given two bounded linear orders X and Y, let $X \boxplus Y$ be their concatenation obtained by gluing $\max X$ and $\min Y$. If $F \colon X \to S$ and $G \colon Y \to S$ are two S-models, we define $F \boxplus G \colon X \boxplus Y \to S$ by

$$(F \boxplus G)(x, y) = \begin{cases} F(x, y) & \text{if } x, y \in X, \\ G(x, y) & \text{if } x, y \in Y, \\ F(x, \max X) G(\min Y, y) & \text{if } x \in X,\ y \in Y. \end{cases}$$

Proposition 4.7. *Let S be a profinite monoid such that P_S is a Δ_{bound}-adic space. Let $F \colon X \to S$ and $G \colon Y \to S$ be two S-models. Then $F \boxplus G$ is also an S-model.*

Proof. We need to prove that if $(F \boxplus G)(x, y) \in UV$ with $U, V \subseteq S$ clopen subsets, $x \in X$ and $y \in Y$, then there is some z in X or some z in Y such that $(F \boxplus G)(x, z) \in U$ and $(F \boxplus G)(z, y) \in V$. Let $a \coloneqq F(x, \max X)$ and $b \coloneqq G(\min Y, y)$. We know that $ab = uv$ for some $u \in U$ and $v \in V$. Because of the amalgamation property, this means that, without loss of generality, there is some $k \in S$ such that $kb = v$ and $uk = a$. We deduce that $a \in U(V/b)$. Hence, there exists some $z \in X$ such that $F(x, z) \in U$ and $F(z, \max X) \in V/b$. We get $(F \boxplus G)(x, z) \in U$ and $(F \boxplus G)(z, y) = F(z, \max X) G(\min Y, y) \in V$ as desired. □

This operation of concatenation can be explained abstractly as follows. Let P, Q be two Δ_{bound}-adic spaces. The theory whose models are pairs of models, one of P and one of Q, is represented by the functor $P \times Q \colon \Delta_{\text{bound}} \times \Delta_{\text{bound}} \to$ **Stone** sending (n, m) to $P(n) \times Q(m)$. As said in example 3.5, left Kan extension along $\boxplus\colon \Delta_{\text{bound}} \times \Delta_{\text{bound}} \to \Delta_{\text{bound}}$ (the functor of concatenation by gluing the endpoints) translates a $\Delta_{\text{bound}} \times \Delta_{\text{bound}}$-adic space as a Δ_{bound}-adic space by forgetting the distinguished point. We will write $P * Q$ for the left Kan extension of $P \times Q$ along $\boxplus$. It is called the *Day convolution* of P and Q and we have the coend formula $[P * Q](n) = \int^{(a,b)} P(a) \times Q(b) \times \mathrm{Hom}(n, a \boxplus b)$.

As in example 3.5 again, if P is a Δ_{bound}-adic space, then the $\Delta_{\text{bound}} \times \Delta_{\text{bound}}$-adic space $(a, b) \mapsto P(a \boxplus b)$ represents the theory whose models are pointed models of P. Let P_* be the left Kan extension of this functor along $\boxplus$. Then P_* is the theory P with a distinguished constant freely added and the counit $P_* \to P$ forgets this new constant. A monoidal functor structure on P is given by an isomorphism $P(a) \times P(b) \cong P(a \boxplus b)$, i.e. an isomorphism $P * P \cong P_*$. By composing $P * P \cong P_* \to P$, we get the concatenation of Proposition 4.7.

If X and Y are two bounded linear orders, thought of as ind-objects $\Delta_{\text{bound}}^{\text{op}} \to \mathbf{Set}$, then their Day convolution $X * Y$ is $X \boxplus Y$. To concatenate two models $X \to P$ and $Y \to P$, we compose $X * Y \to P * P \to P$.

Recall from Proposition 3.10 and remark 3.11 (using Beth's definability theorem) that an isomorphism between Δ_{bound}-adic spaces $P(a) \times P(b) \cong P(a \boxplus b)$ can be seen at the level of models. The theorem below makes this more explicit in the present case.

Theorem 4.8. *Let T be a first-order theory with a bounded linear order symbol. Let $P\colon \Delta_{\text{bound}}^{\text{op}} \to \mathbf{Stone}$ be the corresponding Δ_{bound}-adic space. Suppose that T has exactly one model of cardinality 1, or in other words that $P(\underset{\vdash\!\dashv}{1}) = 1$. Then monoidal functor structures on P are in correspondence with families indexed by pointed bounded linear orders (X, p) of bijections between T-structures on X and pairs of T-structures on the two segments $\{x \in X \mid x \leq p\}$ and $\{x \in X \mid x \geq p\}$, subject to the following conditions:*

1. *The induced concatenation of T-models is associative, or equivalently given a bipointed T-model (X, a, b), the two induced T-structures on $[a, b] \subseteq X$ are equal.*
2. *The unique model of cardinality 1 acts as a neutral element for concatenation.*
3. *Formulas can be relativized to segments: given any formula $\varphi(x_1, \ldots, x_n)$ on n ordered variables ($x_1 \leq \cdots \leq x_n$), there is some formula $\psi(x_1, \ldots, x_n, p)$ on $n+1$ ordered variables such that an $(n+1)$-pointed model $(X, x_1, \ldots, x_n, p)$ of P with $x_1 \leq \cdots \leq x_n \leq p$ satisfies ψ if and only if $(\{x \in X \mid x \leq p\}, x_1, \ldots, x_n)$ satisfies φ. Symmetrically for $p \leq x_1 \leq \cdots \leq x_n$.*

Moreover, it is enough to check the last condition on the symbols of the signature of T other than the order.

Example 4.9. As an example of a profinite monoid built using Theorem 4.8, we can take T to be simply the theory of bounded linear orders. Since the only symbol of the signature is the order, there is almost nothing to check. This monoid is uncountable [15, Corollary 6.17]. If T is instead the theory of successor ordinals, then the monoid becomes countable and has been described in [3, Definition 39].

4.2 Application to logic on words

Logic on words The goal now is to give a presentation of logic on words from the viewpoint of polyadic spaces. Let A be a finite alphabet. Since A^* is equidivisible, the functor $\underline{n} \mapsto [A^*]^n \in \mathbf{Set}$ has the amalgamation property, meaning that it is a Δ_{bound}-adic set. By Property 3.6, the Stone-Čech compactification $\underline{n} \mapsto \beta([A^*]^n)$ is a Δ_{bound}-adic space whose dual is $\underline{n} \mapsto \mathscr{P}([A^*]^n)$. We do not get a profinite monoid this way, but we will obtain some equidivisible profinite monoids as *quotients* of $\beta([A^*]^n)$ in Corollary 4.10.

Let us describe the associated logic. Each finite word over A corresponds to a model whose points are positions between two letters, before the first or after the last one, so that a word with k letters gives a model with $k+1$ elements. A predicate with n free variables is, in the most general possible way, a subset of all the n-pointed finite models. Looking at the corresponding Δ_{bound}-adic space, since $\underset{\vdash\dashv}{n}$-pointed words can be identified with $[A^*]^{n-1}$, a predicate in context $\underset{\vdash\dashv}{n}$ is a subset of $[A^*]^{n-1}$.

As seen in Remark 3.9, each Δ_{bound}-adic set can be decomposed uniquely as a coproduct of irreducible Δ_{bound}-adic sets P with $P(\underset{\vdash\dashv}{2}) = 1$. In the present case, this decomposition gives that $\underline{n} \mapsto [A^*]^n$ is the coproduct of all its finite models. This means that a morphism of Δ_{bound}-adic spaces from $\beta([A^*]^n)$ to $P(\underline{n})$ is a way of interpreting each finite word with k letters as a model of P with $k+1$ points.

The logic associated to a Boolean algebra of languages Let $B \subseteq \mathscr{P}(A^*)$ be a Boolean algebra of languages and make the following assumptions.

1. The morphism $\mathscr{P}(A^*) \to \mathscr{P}(A^* \times A^*)$ dual to multiplication $A^* \times A^* \to A^*$ sends $B \subseteq \mathscr{P}(A^*)$ into $B \sqcup B \subseteq \mathscr{P}(A^* \times A^*)$.[3] This is equivalent to B being included in regular languages and stable by division on the left and on the right, see [5].
2. B is stable under concatenation and contains $\{\varepsilon\}$ where ε is the empty word.

Under condition 1, the whole family $B^{\sqcup n} \subseteq \mathscr{P}([A^*]^n)$ defines a subfunctor and the Stone dual M of B inherits the structure of a profinite monoid, the multiplication being dual to $B \to B \sqcup B$. Condition 2 ensures that the $B^{\sqcup n} \subseteq \mathscr{P}([A^*]^n)$ are stable under the action of the left adjoints. Since hyperdoctrines are algebraic systems (cf. Remark 3.7), the subalgebra $B^{\sqcup n}$ is again a hyperdoctrine. We can also dualize and use Proposition 3.8 instead: quotients of polyadic spaces are still polyadic spaces.

Corollary 4.10. *Under conditions 1 and 2, the dual M of B is an equidivisible profinite monoid with open multiplication, whose neutral element is the only invertible and is topologically isolated. In particular, the free profinite monoid on A is equidivisible.*

[3] The set $B \sqcup B \subseteq \mathscr{P}(A^* \times A^*)$ is the Boolean subalgebra generated by sets of the form $L \times A^*$ and $A^* \times L$ where $L \in B$. It is also the coproduct of two copies of B in the category of Boolean algebras.

By Theorem 4.3, each such Boolean algebra has an associated first-order logic whose predicates in context $\underline{n}$ are the Boolean algebra $B^{\sqcup(n-1)} \subseteq \mathscr{P}([A^*]^{n-1})$. The conditions 1 and 2 ensure that these subsets are stable under the constructions of first-order logic. Here is an example to illustrate more concretely the roles of both conditions.

Example 4.11. Suppose L and L' are two languages in B. Let $[L](-,-)$ be the 2-ary predicate associated to L (formally, it is $A^* \times L \times A^*$). Let $\bot$, $\top$ be the constants for the minimum and the maximum. For instance, $[L](\bot, \top)$ is the formula saying that the whole word is in L. The formula

$$\varphi \coloneqq \forall x : [L](x, \top) \implies \exists y : y \geq x \wedge [L'](\bot, y)$$

expresses that for each suffix in L, there is a prefix in L' ending at least at the start of the suffix. Let us show that the language associated to φ is in B. First, we need to place the formula $[L'](\bot, y)$ in a context where there is one more variable $x \leq y$. This is done using condition 1: there exists a finite list $(U_i, V_i)_{i=1}^n \subseteq B \times B$ such that we can rewrite $[L'](\bot, y)$ as $\bigvee_{i=1}^n [U_i](\bot, x) \wedge [V_i](x, y)$. We then apply the quantifier $\exists y$ using condition 2 and combine with "$[L](x, \top) \implies$", this gives $[\overline{L}](x, \top) \vee \bigvee_{i=1}^n [U_i](\bot, x) \wedge [V_i A^*](x, \top)$. In order to apply the universal quantifier, we negate the formula, put it in normal disjunctive form, apply the existential quantification and negate again to obtain the final formula

$$\bigcap_{s:n \to \{0,1\}} \overline{\Big(\bigcap_{i \in s^{-1}(0)} \overline{U_i}\Big)\overline{\Big(L \cap \bigcap_{i \in s^{-1}(1)} \overline{V_i A^*}\Big)}}.$$

Example 4.12. Here are some Boolean algebras satisfying conditions 1 and 2.

1. The Boolean algebra of all regular languages. The associated logic is Büchi's monadic second-order logic on words.
2. The Boolean algebra of star-free languages. In [6], van Gool and Steinberg apply model theory to study the free pro-aperiodic monoid A. It is built as a submonoid of another monoid $\Lambda(A)$, which is constructed in our setting by applying Theorem 4.8 to the theory of discrete orders with endpoints whose pairs of consecutive points are labeled by A. Since each finite word is in particular a discrete labeled order, we have a morphism of polyadic spaces $\beta([A^*]^n) \to \Lambda(A)^n$ and the free pro-aperiodic monoid is the image of this morphism. We can interpret [6, Lemma 3.4, p. 13] as saying that this image is a polyadic subspace.
3. The Boolean algebra of languages of generalized star-height n.
4. Other examples can be found in [1, Section 3], where one can find a characterization of the pseudovarieties recognizing Boolean algebras of languages satisfying condition 2 (condition 1 is automatically satisfied).

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